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CGLMP and Bell–CHSH formulations of non-locality: a comparative study

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Abstract The CGLMP prescription of non-locality is known to identify certain states that satisfy the Bell–CHSH inequality to be non-local. The demonstration is, however, restricted to a specific family of states. In this paper, we address the converse question: can there be states that satisfy the CGLMP inequality but violate Bell–CHSH? We find the answer to be in the affirmative. Examining coupled 4×4 level systems, we find that there exist a large number of such states. As a direct consequence, states that violate the CGLMP inequality do *not* form a superset over the ones that violate Bell–CHSH.

Keywords Bell inequality · Non-locality · Higher-dimensional systems

1 Introduction

By virtue of their construction, all Bell-type inequalities are obeyed by the set of all local states. Hence, a violation of any of these inequalities is an unmistakable signature of non-locality. The converse statement is, however, not guaranteed to be true. That is, a state that obeys a given Bell-type inequality need not be local. Thus, it is pertinent

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to look at the respective domains of validity and the mutual compatibility of multiple prescriptions of non-locality.

Two such inequivalent formulations of non-locality are Bell–CHSH [3,7] and CGLMP [8,12,15]. The former involves a single inequality with two dichotomic observables at each site. The latter is a dimension-dependent inequality also with two observables per site but each having distinct eigenvalues. CGLMP reduces to Bell–CHSH for two qubits. By applying the CGLMP prescription to the family of fully entangled states $|\Psi_E\rangle = (1/\sqrt{N})\sum_{j=1}^{N}|jj\rangle$, in $N \times N$ dimensions, it was found that the violation is always greater than $2\sqrt{2}$ for $N \geq 3$, which becomes larger with increasing dimensions. As a direct consequence, when $|\Psi_E\rangle$ is contaminated with white noise,

$$\rho = p|\Psi_E\rangle\langle\Psi_E| + (1-p)\frac{\mathbb{I}}{N^2}, \tag{1}$$

the family of states (as characterised by the range of p) that violate CGLMP can be seen to be larger than the ones that violate Bell–CHSH when $N \geq 3$. Based on this result, it was concluded in [8,15] that CGLMP can indeed identify non-local states that evade the Bell analysis. The predicted violations have been subsequently verified in a series of experiments for maximally entangled states [9,14,16,21,22], the last one involving coupled systems up to $N = 16$.

From this example, it would appear that CGLMP is more efficient in detecting non-local states than Bell–CHSH. If true, the states that violate CGLMP could form a superset over the ones that violate Bell–CHSH. However, substantiation of such a statement requires that we do not restrict ourselves to testing the violation of CGLMP for any particular class of states. This paper undertakes this task of examining whether CGLMP is more discriminating of non-local states than the Bell–CHSH inequality. Conversely, we explore the possibility of the existence of Bell–CHSH non-local states that satisfy the CGLMP inequality. To that end, we test CGLMP against the family of *all* Bell states, i.e. those that violate Bell–CHSH inequality maximally as well as the family of those states that violate the Bell–CHSH inequality marginally, in coupled four-level systems.

2 Formulation

2.1 Bell–CHSH and CGLMP inequalities

For the sake of completeness, we describe the CGLMP and the standard Bell–CHSH inequalities briefly. Consider an $M \times N$ level system with two subsystems A and B of M and N levels, respectively. The Bell operator $\mathcal{B}$ is defined by,

$$\mathcal{B} = A_1B_1 - A_1B_2 + A_2B_1 + A_2B_2, \tag{2}$$

where $A_{1,2}$ and $B_{1,2}$ are dichotomic observables for the subsystems A and B, respectively. They satisfy the conditions $-1 \leq \langle A_i \rangle \leq 1$ and $-1 \leq \langle B_i \rangle \leq 1$. Local hidden variable models constrain the Bell function to obey the inequality $|\langle\mathcal{B}\rangle| \leq 2$, a violation of which implies non-locality. As a non-local theory, quantum mechanics pushes the upper bound to a value $2\sqrt{2}$ [6]. This bound is absolute and independent of M and N.

CGLMP inequality has a more complicated structure. The analog of the Bell operator is the function I_N, defined for an $N \times N$ level system by

$$I_N = \sum_{k=0}^{[N/2]-1} \left(1 - \frac{2k}{N-1}\right) \{[P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) \\ + P(A_2 = B_2 + k) + P(B_2 = A_1 + k)] \\ - [P(A_1 = B_1 - k - 1) + P(B_1 = A_2 - k) \\ + P(A_2 = B_2 - k - 1) + P(B_2 = A_1 - k - 1)]\}. \tag{3}$$

where $P(A_i, B_i)$ are joint measurement probabilities for local observables A_i, B_i belonging to the two subsystems. All the observables have distinct integer eigenvalues $0, 1, \ldots, N-1$. The measurement prescription can be found in [10] and involves two local observers, Alice and Bob, who fine-tune variable phases α_i, β_i (see Eq. 4) of the states in their respective subsystems, depending on the measurements they wish to perform. The measurement bases for the observables A_i and B_i, $i = 1, 2$, are mutually unbiased and of the form:

$$|K\rangle_{A,i} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp\left(i \frac{2\pi}{N} j (K + \alpha_i)\right) |j\rangle_A, \\ |L\rangle_{B,i} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp\left(i \frac{2\pi}{N} j (-L + \beta_i)\right) |j\rangle_B. \tag{4}$$

Rules of classical probability impose the constraint $|I_N| \leq 2$. This constraint gives the locality condition inasmuch as it arises in measurements involving joint probabilities.

On evaluating I_N for the maximally entangled state, it is found that the maximum value of I_N as allowed by quantum theory overshoots the Tsirelson bound when $N \geq 3$. It takes a value 2.8962 when $N = 4$, and approaches the limiting value of 2.9696 as $N \to \infty$. Since $I_4 > 2\sqrt{2}$ for the Bell state, it follows that some noisy states (defined in Eq. 1) will obey Bell–CHSH inequality but violate CGLMP. As remarked, experiments are also performed on maximally entangled states.

3 Contrasting CGLMP and CHSH inequalities

We wish to contrast the CGLMP inequality with the Bell formulation for a broader class of states for a 4×4 level system. Our first test employs Bell states, that is, those that violate Bell–CHSH maximally. In this, we freely exploit the abundance of Bell states which are not restricted to be fully entangled, or be pure. We follow the analysis in [4] closely in the construction of Bell states. We also test the two inequalities for a class of states that violate Bell–CHSH marginally.

3.1 Bell states of 4 × 4 level systems

It is known that the conditions on the local observables for attaining the Tsirelson bound are given by [19]

$$\left\langle A_{1,2}^2 \right\rangle = \left\langle B_{1,2}^2 \right\rangle = 1$$
$$\left\langle \{A_1, A_2\} \right\rangle \text{ or } \left\langle \{B_1, B_2\} \right\rangle = 0. \tag{5}$$

The two conditions jointly constitute the definition of Clifford Algebra, the representations of which are essentially given by the standard Pauli matrices or their direct sums for each pair of observables. This result is a consequence of Jordan's theorem [20]. It follows thereof that maximally non-local Bell states are either coherent, or incoherent superpositions of Bell states in mutually orthogonal 2×2 sectors. This explains why there are no fully entangled Bell states when N is odd.

Armed with this result we conveniently choose the observables to be

$$A_1 = \frac{2}{\sqrt{3}}\lambda_8 + \frac{\sqrt{6}}{3}\lambda_{15}; \quad A_2 = (\lambda_4 + \lambda_{11}),$$
$$B_1 = \frac{1}{\sqrt{2}}(A_1 + A_2); \quad B_2 = \frac{1}{\sqrt{2}}(A_2 - A_1), \tag{6}$$

where the $SU(4)$ generators (the λ matrices) are taken in their standard form. Note that the observables are all dichotomic: The spectrum of the Bell operator is easily determined. It has the eigenresolution

$$\mathcal{B} = 2\sqrt{2}(\Pi_{\mathcal{H}_+} - \Pi_{\mathcal{H}_-}), \tag{7}$$

where $dim(\mathcal{H}_\pm) = 4$. The bases for $\mathcal{H}_\pm$ may be chosen to be

$$|\eta_1\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |33\rangle); \quad |\eta_2\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |32\rangle),$$
$$|\eta_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |23\rangle); \quad |\eta_4\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |22\rangle), \tag{8}$$

and

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|31\rangle - |13\rangle); \quad |\phi_2\rangle = \frac{1}{\sqrt{2}}(|30\rangle - |12\rangle),$$
$$|\phi_3\rangle = \frac{1}{\sqrt{2}}(|21\rangle - |03\rangle); \quad |\phi_4\rangle = \frac{1}{\sqrt{2}}(|20\rangle - |02\rangle), \tag{9}$$

respectively. Within each sector, all states, both pure and mixed, violate the Bell–CHSH inequality maximally. Thus, in contrast to the two-qubit case, Bell states can have ranks ranging from 1 to 4. We consider states belonging to $\mathcal{H}_+$ henceforth. We examine the pure and mixed states separately in the next section.

3.2 Comparison of Bell–CHSH and CGLMP measures

3.2.1 Pure states

The comparison requires a numerical study of the behaviour of I_4. It is convenient to represent a pure Bell state in the form

$$|\Psi\rangle_{\mathcal{H}_+} = \sum_i c_i\,|\eta_i\rangle \;\; ; \;\; \sum_i |c_i|^2 = 1, \tag{10}$$

with the parametrisation

$$\begin{aligned} c_1 &= \cos\theta_1, \\ c_2 &= \exp(i\gamma_1)\sin\theta_1\cos\theta_2, \\ c_3 &= \exp(i\gamma_2)\sin\theta_1\sin\theta_2\cos\theta_3, \\ c_4 &= \exp(i\gamma_3)\sin\theta_1\sin\theta_2\sin\theta_3. \end{aligned} \tag{11}$$

The states constitute a six dimensional manifold $\mathcal{M}^6 \equiv \mathcal{S}^3 \times (\mathcal{S}^1)^{\otimes 3}$. The task consists of optimising the tunable phases α_i, β_i (Eq. 4) in order to maximise I_4. For instance, numerical simulations performed in [10] determine the optimal values for the maximally entangled state $|\Psi_E\rangle$ to be $(\alpha_1, \alpha_2) = (0, 1/2)$; $(\beta_1, \beta_2) = (1/4, -1/4)$. This configuration yields a value of $I_4 = 2.8962$, which is in excess of $2\sqrt{2}$.

The method In order to find the maximum I_4 value for each state, we implement the Nelder–Mead optimisation technique [18], to search over the four-dimensional parameter space spanned by the phases $\{\alpha_{1,2}, \beta_{1,2}\}$. The same technique was used in [10,15] to optimise the parameters for the fully entangled state. This method does not always guarantee convergence; the search stops when the 'standard error' falls under a certain pre-defined value of the standard deviation, $\sqrt{\frac{1}{n+1}\sum_{i=0}^{n}(f(x_1) - \overline{f(x_i)})^2}$. Its success depends on the simplex not becoming too small in relation to the curvature of the surface. If the simplex is too small, chances of it being trapped in a local extrema are high. However, through selection of different initial test points across the parameter space to commence the search, one can avoid repeatedly falling into the same local extrema, and a global extrema can be obtained.

For each state in our simulations, a number of such searches were performed, and the numerical results obtained in [10,15] verified. Henceforth, we simply denote I_4 as the maximum value pertaining to each state across these multiple searches. 1000 pure states were sampled randomly from $\mathcal{M}^6$ through uniform distributions. The pre-defined error criterion was taken to be 0.0001. The results are given in the histogram in Fig. 1.

We see that most of the Bell states respect the CGLMP inequality. Out of the 1000 states, it is found that only 8.9% violate the CGLMP inequality. A polynomial fit shows the population of the sampled states decreasing as a function of I_4, at a rate $(I_4)^{-x}$, where $x = 3.66 \pm 0.49$ (see Fig. 1). This reflects the sparsity of states that

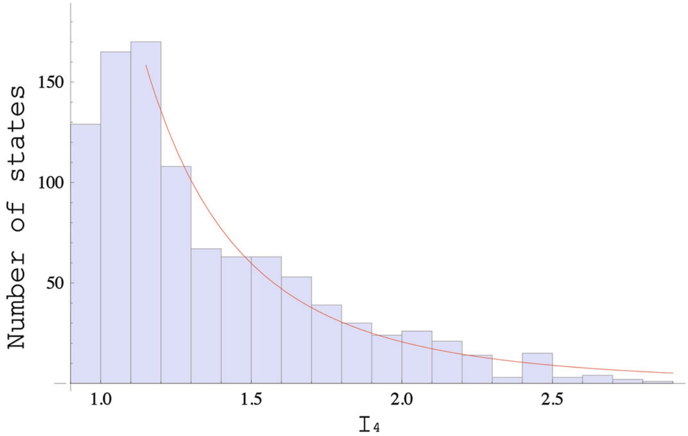

Fig. 1 (colour online) CGLMP I_4 values for 1000 randomly sampled pure states over the manifold of maximally Bell–CHSH non-local states. Histogram bin width: 0.1. Red: polynomial fit showing population decay. Black: $I_4 = 2$, demarcation between local and non-local states

obey the CGLMP criterion for non-locality in contrast to the Bell–CHSH criterion. In fact, there is only one state with $I_4 > 2\sqrt{2}$ from among the 1000 random states.

3.2.2 Mixed states

The space of mixed states is much larger, being of dimension 15. We examine the rather small subset of states which are given by:

$$\rho_{\mathcal{H}_+} = \sum_i p_i |\eta_i\rangle\langle\eta_i|. \tag{12}$$

Once again, we find that the CGLMP prescription fails to identify non-locality, this time more dramatically. We sample 100 random mixed states and implement the Nelder–Mead optimisation technique, in the same manner that was done for pure states. Figure 2 shows the distribution of states over I_4 values. All of these states have values of I_4 very close to zero, despite being maximally Bell–CHSH non-local.

3.2.3 States violating Bell–CHSH inequalities marginally

We now turn our attention to the set of states that violate Bell–CHSH marginally. We know that any partially entangled pure state necessarily violates the Bell–CHSH inequality [13]. Consider the following family of states:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{1-\epsilon^2}\,|11\rangle + \epsilon\,|33\rangle\right) + \frac{1}{\sqrt{2}}\left(\sqrt{1-\epsilon^2}\,|10\rangle + \epsilon\,|32\rangle\right). \tag{13}$$

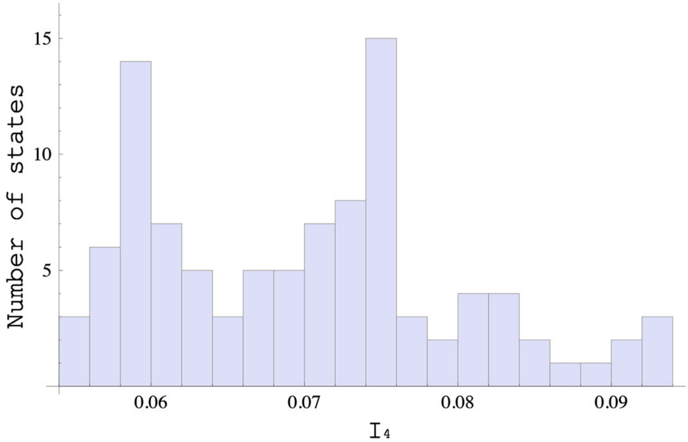

Fig. 2 (colour online) CGLMP I_4 values for 100 randomly sampled mixed states over the manifold of maximally Bell–CHSH non-local states. Histogram bin width: 0.002

We find that for $\epsilon = 0, 1$ the state becomes separable. We hence restrict our study to the neighbourhoods of these two ϵ values, which contain slightly entangled pure states with a marginal Bell–CHSH violation.

Using a uniformly distributed random number generator, we generated 10 states each from the neighbourhoods defined by $\epsilon \in (0.005, 0.05)$ and $\epsilon \in (0.95, 0.99)$. Using the aforementioned Nelder–Mead technique, we evaluated the I_4 values for each of these 20 states for not only the CGLMP form provided by Eq. 3, but its equivalent forms as well, as outlined in the following subsection. None of these marginally Bell–CHSH non-local states violated the CGLMP inequality.

Yet again our example proves the existence of Bell–CHSH non-local states that obey the CGLMP inequality. We also find that the population of CGLMP non-local states bears no affinity to the Bell–CHSH violations and turns out to be equally sparse in both the case studies (maximally and marginally Bell–CHSH violating states).

3.2.4 Equivalent forms

CGLMP has a number of other (equivalent) forms. They can be obtained from Eq. 3 following a set of three independent transformations, *viz.* (i) party exchange ($P(A_a = x, B_b = y) \rightarrow P(A_b = y, B_a = x)$), (ii) observable exchange ($P(A_a = x, B_b = y) \rightarrow P(A_{\bar{a}} = x, B_b = y)$) and (iii) relabelling of outcomes ($P(A_a = x, B_b = y) \rightarrow P(A_a = x \oplus i, B_b = y)$), where i is an integer from 0 to 3 [17]. For the sake of completeness, we have also tested for violations of these transformed inequalities, for 100 Bell states, randomly sampled out of the 1000 pure Bell states used in Sect. 3.2.1. Again, most of the states still evade any violation of CGLMP in all the three cases.

It may therefore be concluded that the *apparent advantage* of CGLMP over Bell–CHSH in terms of detecting non-local states cannot be substantiated, unless perhaps the states are highly entangled. Also, the set of states that violate the CGLMP inequality is far from forming a superset over the set that violates Bell–CHSH.

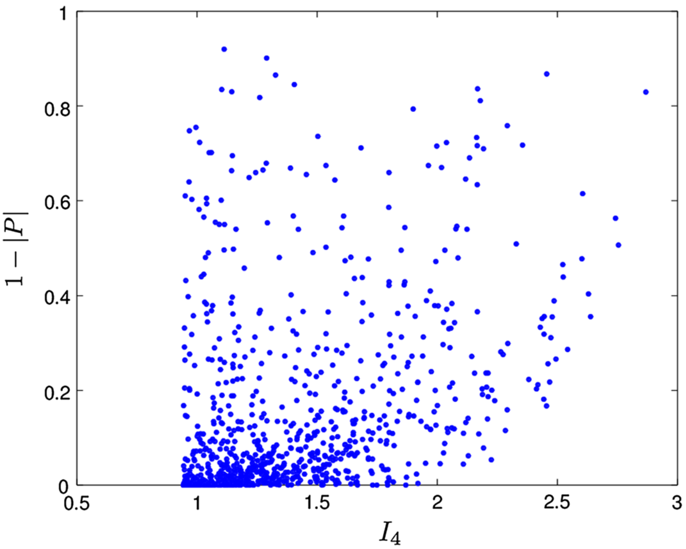

Fig. 3 (colour online) CGLMP I_4 plotted as a function of entanglement, $(1 - |\mathbf{P}|)$, where $\mathbf{P}$ is the Bloch vector in Eq. 14

Note that this failure gets carried over to large classes of Bell states in all higher dimensions in a more pronounced manner. Since it fares well for fully entangled states, we examine whether I_4 is more sensitive to entanglement of the state.

3.2.5 CGLMP and entanglement

We consider pure Bell states. Their reduced states of the subsystem have a block diagonal form with identical 2×2 matrices. This simple structure allows us to devise a measure of entanglement that is much simpler and yet completely equivalent to the von Neumann entropy. Each of the sectors may be considered as a pseudo-qubit sector, which is completely defined by a Bloch vector $\mathbf{P}$. The resulting eigenvalues of the reduced density matrix are, therefore, twofold degenerate and can be written as

$$\mu_{\pm} = \frac{1}{4}(1 \pm |\mathbf{P}|); \quad 0 \le |\mathbf{P}| \le 1. \tag{14}$$

The quantity $1 - |\mathbf{P}|$ is a measure of entanglement and is a relative monotone of the standard von Neumann entropy. The value $|\mathbf{P}| = 0$ represents a fully entangled state, and $|\mathbf{P}| = 1$, a partially entangled state with the corresponding entropy of the reduced density matrix being $\log 2$.

Figure 3 shows the variation of I_4 with respect to $1-|\mathbf{P}|$ for the 1000 states employed earlier. The scatter in the plot clearly shows that I_4 bears no affinity to entanglement. In fact, it has been shown for coupled three-level systems that the CGLMP violation for certain non-maximally entangled states may exceed that of the maximally entangled one [1]. An alternative measure of non-locality that is monotonic with entanglement has been proposed in [11].

4 Discussion and conclusion

The main result of this paper is an explicit demonstration of the existence of non-local states that violate the Bell–CHSH inequality but evade the CGLMP analysis. Though CGLMP works better as a discriminator of non-locality along the Werner line, it falls short for a large family of less entangled states vis-a-vis Bell–CHSH. This rules out the possibility of the CGLMP violating states forming a superset over the Bell–CHSH violating states.

In conclusion, we find that CGLMP and Bell–CHSH are two independent criteria that get satisfied by different sets of local states. Satisfaction of either inequality does not guarantee locality. An unequivocal detection of local states would require an evaluation of all facet inequalities defined over the local polytope [2,5].

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